A High-Order Accurate Unstructured GMRES Solver for the Compressible Euler Equations

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Abstract

A preconditioned matrix-free GMRES method is presented for unstructured higher-order computation of 2D Euler equations. Lower-Upper Symmetric Gauss-Seidel (LU-SGS) has been used as the preconditioning strategy, and a first-order Jacobian as the preconditioner. The numerical algorithm demonstrates promising convergence performance for both the second and the third order discretization methods in supersonic flow.

1 Introduction

Recent research on structured mesh flow solver for aerodynamics shows that for practical levels of accuracy using a higher order accurate method can reduce computation time and memory usage and improve robustness compared to a second-order scheme [7, 15]. To take advantage of the high flexibility in mesh generation and adaptation for unstructured meshes, we want to apply higher-order accurate methods for unstructured meshes. This approach combines benefits of higher-order methods and unstructured meshes. Although high-order accurate methods for unstructured meshes are reasonably well established [1, 2, 6, 10], application of these methods for physically complicated flows is still a challenge due to very slow convergence. This eliminates the efficiency benefits of higher-order unstructured discretization. Consequently, convergence acceleration becomes the key issue for the practical usage of higher-order unstructured solvers. Recent results of an unstructured mesh solver for Poisson's equation [9] clearly showed the possibility of reducing computational cost required for a given level of solution accuracy using higher-order methods and matrix free GMRES [13] as a convergence acceleration technique. As Poisson's equation is simple and has a linear flux, the resultant Jacobian matrix is well-conditioned, and the convergence rate for GMRES technique is good. However, as GMRES convergence is sensitive to the condition number of Jacobian matrix, in the case of complex equations (such as Euler) which normally have an ill-conditioned Jacobian, employing an effective preconditioner for GMRES technique becomes a key necessity. For structured meshes, Pueyo and Zingg [11] presented an efficient matrix-free Newton-GMRES solver using multi-grid for steady aerodynamic flows.

Barth and Linton [3] successfully applied both full-matrix and matrix-free GMRES iteration for computing compressible fluid flow over unstructured meshes. Blanco and Zingg [4] compared the use of two variations on quasi- and full-Newton method for solution of the Euler equations on unstructured grids. Recently Manzano et. al. [8] developed a preconditioned matrix-free GMRES method to solve 2-D and 3-D Euler equations using ILU preconditioner based on an approximation to the flow Jacobian.

Our objective in this research is to develop an efficient and accurate unstructured solver for inviscid compressible fluid flow. This paper presents preliminary results of using LU-SGS for preconditioning of the matrix-free GMRES to compute the higher-order solution of Euler equations.

2 Governing Equations

The unsteady (2D) Euler equations which model compressible inviscid fluid flows, are conservation equations for mass, momentum, and energy. The finite-volume formulation of Euler equations for an arbitrary control volume can be written in the following form of a volume and a surface integral.

$$\frac{d}{dt}\int_{cv} \boldsymbol{U} dv + \oint_{cs} \boldsymbol{F} dA = 0 \tag{1}$$

where

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} \rho u_n \\ \rho u u_n + P\hat{n}_x \\ \rho v u_n + P\hat{n}_y \\ \rho(E+P)u_n \end{bmatrix} \tag{2}$$

In (2), $u_n = u\hat{n}_x + v\hat{n}_y$ and $(\rho \quad \rho u \quad \rho v \quad E)^T$ are the densities of mass, x-momentum, y-momentum, and energy, respectively. The energy is related to the pressure by the perfect gas equation of state: $E = P/(\gamma - 1) + \rho(u^2 + v^2)/2$, with γ the ratio of specific heats for the gas.

3 Numerical Algorithm

Linearization of the governing equations in time and applying implicit time integration leads to implicit time advance formula:

$$\left(\frac{I}{\Delta t} + \frac{\partial R}{\partial U}\right)\delta U = -R_i \qquad , \qquad \delta U_i = U_i^{n+1} - U_i^n \tag{3}$$

where $\partial R/\partial U$ is the Jacobian matrix. To advance the solution from one time step to the next and eventually converging to the steady-state solution, a large system of linear equations must be solved. The linear system is asymmetric both in fill and values making GMRES the obvious first choice in iterative solvers. The linear system arising from high-order discretization has four-five times as many non-zero entries as a second-order scheme. Because of the size of these matrices and the difficulty in computing their entries analytically (even for the second order), we have chosen to use an entirely matrix-free implementation of GMRES. Since the GMRES algorithm only needs matrix-vector products, and these products can be computed by matrix-free approach [3], matrix-free GMRES is a very attractive technique for dealing with the complicated Jacobian matrices resulting from higher-order discretization. However, as GMRES convergence is highly dependent on the condition number of Jacobian matrix, in the case of Euler with nonlinear flux function and possible discontinuities in the solution, using a preconditioner for GMRES becomes necessary for practical purposes. At the same time, using high-order discretization makes the Jacobian matrix more off-diagonally dominant and quite ill-conditioned. Therefore, we use flexible GMRES known as FGMRES [14] with LU-GSG [5] as a preconditioning strategy. To build the preconditioner, we form a first-order Jacobian matrix, reducing the size and complexity of the matrix.

In our discretization scheme, a higher-order accurate least-square reconstruction procedure [10] has been used in the interior of the domain in order to compute fluxes using Roe's difference-splitting scheme [12] at control volume boundaries up to the desired accuracy. Having computed the fluxes, we integrate along each control volume boundary using Gauss quadrature integration technique with the proper number of points to get the required order of accuracy for the flux integral. Imposing the boundary conditions also has been done to high-order accuracy.

4 Results

To investigate the convergence performance and robustness of the purposed GMRES solver with a higher order unstructured discretization, different supersonic flow cases have been studied. Here for brevity, we only present one of them which shows the general convergence behavior of our numerical experiments. Our test case (Fig.1) is a 5m supersonic duct with an expansion corner located 1m after inlet at the lower wall and the corner angle is 9.23°. The height of the inlet and outlet are 0.75m and 1.4m respectively. 2100 control-volumes are used in this test case with a proper refinement at expansion corner, and inlet Mach number is equal to 2. As convergence performance and stability of GMRES technique especially for compressible flows are quite sensitive to the start-up process and the solution at initial iterations, we have to do several implicit iterations before switching to GMRES iteration. For both the second and third order discretization, 20 implicit iterations have been performed before GMRES iteration. The iterations start with $CFL = 1.0$ and CFL number is increased gradually to 100. For the fourth order discretization, we do 275 pre-iterations with similar CFL number increasing trend.

The residual convergence history in terms of number of iterations and residual evaluations for the test case has been shown in Fig.2 and Fig. 3. The figures show considerable drop in residual as GMRES iteration starts. GMRES-LUSGS effectively decreases the residual (10 orders of magnitude) by about 15 iterations for the second order and about 35 iterations for the third order discretization. However, for the fourth order discretization, GMRES-LUSGS loses its quadratic convergence behavior after 15 iterations and the convergence slows considerably. The convergence result shows 10 order drop in L2-norm (density residual) by about 1200 and 2700 residual evaluations for the second and the third order cases. Also 9 order residual reduction has been achieved by 13000 residual evaluations for 4th order discretization by GMRES-LUSGS(Fig.3) . To demonstrate the effect of (LU-SGS) preconditioning on GMRES, the residual convergence histories for the simple GMRES without preconditioner have been shown in similar graphs for all different orders of accuracy, and as shown using a preconditioner has improved the performance and robustness of GMRES technique noticeably.

5 Concluding Remarks

A preconditioned matrix-free LUSGS-GMRES algorithm has been presented for higher-order computation of solution of 2D Euler equations. The results show that LUSGS-GMRES works almost as efficiently for the third order discretization as for the second order one. In the case of fourth order discretization, still there are challenges ahead, and as a future work we will concentrate on improving current approach especially for the 4th order accurate unstructured mesh discretization.

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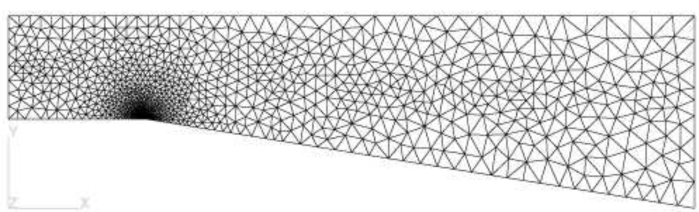

Fig. 1. Domain and mesh for the supersonic duct

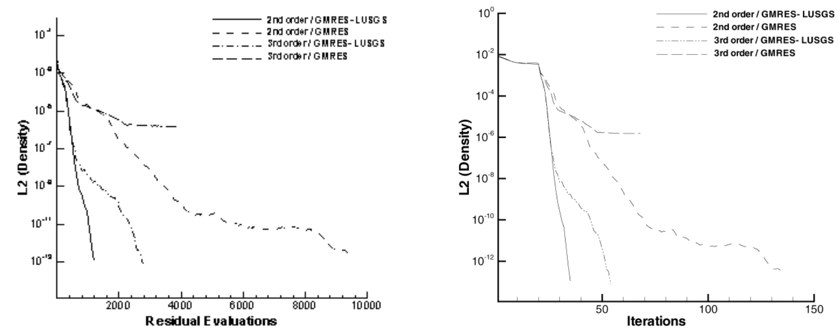

Fig. 2. Convergence history for the second and third order methods

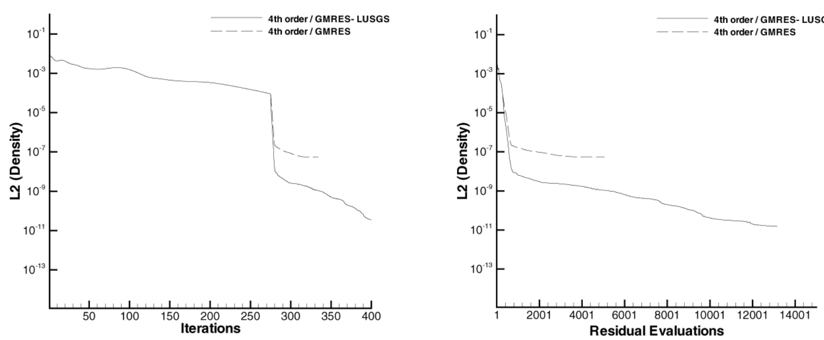

Fig. 3. Convergence history for the fourth order method

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